Almost Lehmer numbers

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Introduction

Let

$\varphi(n)$: the Euler totient of n, the number of positive integers $d \leq n-1$ coprime to n.

Clearly, $\varphi(n) = n - 1$ if and only if n is prime.

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Further results

Cohen and Hagis, 1980: $\omega(n) \geq 14$ and $n > 10^{20}$.

Renze's notebook: $\omega(n) \geq 15$ and $n > 10^{26}$.

Pinch claims at his research page: $n > 10^{30}$.

Moreover, letting $V(x)$ be the number of composites $n \leq x$ such that $\varphi(n) \mid (n-1)$,

Pomerance, 1977: $V(x) = O(x^{1/2} \log^{3/4} x)$ and $n \leq r^{2^r}$ if $2 \leq \omega(n) \leq r$ additionally.

Luca and Pomerance, 2011: $V(x) < x^{1/2} \log^{-1/2+o(1)} x$.

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The first few composite 2-Lehmer numbers:

$$561, 1105, 1729, 2465, \ldots$$

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Following estimates are known:

McNew, 2013

For each k, the number of k-Lehmer composites is $O(x^{1-1/(4k-1)})$ and the number of integers which are k-Lehmer composites for some k is at most $x \exp(-(1+o(1)) \log x \log\log\log x / \log\log x)$.

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For each $k \geq 3$, there exist at least $x^{1/(k-1)+o(1)}$ integers $n \leq x$ which are k-Lehmer but not $(k-1)$-Lehmer numbers.

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Nearly and almost Lehmer numbers

Now we would like to discuss intermediate properties between the 1-Lehmer (that is, ordinary Lehmer) property and 2-Lehmer property.

Almost Lehmer numbers

We call an integer n to be

(a) an almost Lehmer number if $\varphi(n)$ divides $\ell(n-1)$ for some squarefree divisor ℓ of $n-1$, and

(b) an r-nearly Lehmer number if $\varphi(n)$ divides $\ell(n-1)$ for some squarefree divisor ℓ of $n-1$ with $\omega(\ell) \leq r$.

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Instances

- The ordinary Lehmer property is equivalent to the 0-nearly Lehmer property and an almost Lehmer numbers can be regarded as ∞-nearly Lehmer numbers.
- The first few almost Lehmer composites are

$$1729, 12801, 247105, 1224721, 2704801, 5079361, 8355841, \ldots,$$

  given in A337316.
- There exist exactly 38 almost Lehmer composites below 2^{32}.
- There exist only five 1-nearly Lehmer composites 1729, 12801, 5079361, 34479361, and 3069196417 below 2^{32} (further instances are given in the discussion of A338998).

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We use the following notion:

- U_r: the set of composites n for which $\varphi(n)$ divides $\ell(n-1)$ for some squarefree divisor ℓ of $n-1$ with $\omega(\ell) \leq r$.
- Thus, U_∞ denotes the set of almost Lehmer composite numbers.
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Main results

Theorem 1 (Y., submitted)

Let a_r be the number of partitions of the multiset $\{1, 1, 2, 2, \ldots, r, r\}$ of r integers repeated twice. Then, there exist two absolute constants c and c_1 such that, for each $r \geq 1$,

$$\#U_r(x) < ca_r(x \log x)^{2/3}(c_1 \log \log x)^{2r+2/3}. \tag{2}$$

Theorem 2 (Y., submitted)

$$\#U_\infty(x) < x^{4/5} \exp\left(\left(\frac{4}{5} + o(1)\right) \frac{\log x \log \log \log x}{\log \log x}\right), \tag{3}$$

where $o(1) \to 0$ as $x \to \infty$.

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$$\#U_\infty(x) < x^{4/5} \exp\left(\left(\frac{4}{5} + o(1)\right) \frac{\log x \log\log\log x}{\log\log x}\right), \tag{3}$$

where $o(1) \to 0$ as $x \to \infty$.

The first terms of a_r's are

$$2, 9, 66, 712, 10457, 198091, 4659138, 132315780, \ldots$$

given in A020555 and Bender's asymptotic formula (Bender, 1974) yields that

$$\log a_r < 2r\left(\log(2r) - \log\log(2r) - 1 - \frac{\log 2}{2} + o(1)\right) \quad (4)$$

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Hence, setting c and c_1 as in Theorem 1, we obtain

$$\#U_1(x) < 2c(x \log x)^{2/3}(c_1 \log\log x)^{2r+2/3} \tag{5}$$

and

$$\#U_r(x) < c\left(\frac{(e\sqrt{2}+o_r(1))r}{\log r}\right)^{2r}(x \log x)^{2/3}(c_1 \log\log x)^{2r+2/3}, \tag{6}$$

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On the other hand, the above instances lead us to:

Conjecture

There exist infinitely many almost Lehmer composite numbers.

Moreover, there may be infinitely many 1-nearly Lehmer composite numbers (it may occur that $\#U_1(x) \gg \log x$), although such integers are distributed very rarely below our search limit. However, these also seem to be difficult to prove or disprove; it is even not known whether there exist infinitely many 2-Lehmer numbers or not!

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Preliminary Lemmas

Let $\tau(s)$ be the number of multiplicative partitions / factorizations of $s = s_1 s_2 \cdots s_r$ with $s_1 \leq s_2 \leq \cdots s_r$.
The values of $\tau(s)$ for positive integers s are given in A001055.

If $s = p_1^2 p_2^2$, then there exist nine factorizations: $\{p_1^2 p_2^2\}$, $\{p_1^2 p_2, p_2\}$, $\{p_1 p_2^2, p_1\}$, $\{p_1^2, p_2^2\}$, $\{p_1^2, p_2, p_2\}$, $\{p_2^2, p_1, p_1\}$, $\{p_1 p_2, p_1 p_2\}$, $\{p_1 p_2, p_1, p_2\}$, $\{p_1, p_1, p_2, p_2\}$.

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$$\sum_{q\le x, q\equiv 1 \pmod{s}} \frac{1}{q} < \frac{c_1 \log\log x}{s} \tag{7}$$

with some absolute constant c_1, where q runs over all primes satisfying $q \le x, q \equiv 1 \pmod{s}$.

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Lemma 1

For each integer $s \geq 1$, let $S(s;x)$ denote the set of positive integers $n \leq x$ such that s divides $\varphi(n)$. Then

$$S(s;x) \leq \frac{\tau(s)x(c_1 \log\log x)^{\Omega(s)}}{s}, \tag{9}$$

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We observe that if $s \mid \varphi(n)$, then we can take a factorization of $s = s_1 s_2 \cdots s_{t+1}$ with $1 < s_1 < s_2 < \cdots s_t$ such that:

- $q_i \equiv 1 \pmod{s_i}$ for $i = 1, 2, \ldots, t$,
- s_{t+1} divides $q_1^{f_1-1} q_2^{f_2-1} \cdots q_t^{f_t-1}$, and
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For each factorization $s = s_1 s_2 \cdots s_{t+1}$, the number of such integers $n \le x$ does not exceed

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Lemma 2

As x tends to infinity, we have

$$\sum_{s \leq x} \frac{\tau(s)}{s} < \frac{(1+o(1))e^{2\sqrt{\log x}} \log^{1/4} x}{2\sqrt{\pi}}. \quad (10)$$

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Proof of main results

r: a positive integer or ∞,

x: a sufficiently large real number ,

n: be an r-nearly Lehmer number $\leq x$ which is composite.

Clearly, we can write $(n-1)/\varphi(n) = k/\ell$, where

k and ℓ: coprime integers,

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* Now let $L_1 > x^{1/3}$ and $L_2 = L_1^2$ be real numbers which will be chosen later in different manners according to whether r is an integer or $r = \infty$.
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For each d, the number of integers $n = md \le x$ satisfying (12) is at most $1 + \lfloor \ell_2 x/(d\varphi(d)) \rfloor$.

We note that $\ell_2 \le \sqrt{\varphi(d)} \le L_1$.

$d/\varphi(d) < (e^\gamma + o(1)) \log\log d \ll \log\log x$ from, for example, Theorem 328 of Hardy-Wright.

Hence,

$$\begin{aligned} \#U_r(x) &\le \sum_{\ell_2 \le L_1} \sum_{L_1 \le d \le L_2, \ell_2^2 \mid \varphi(d)} \left(1 + \frac{\ell_2 x}{d\varphi(d)}\right) \\ &\ll \sum_{\ell_2 \le L_1} \left(\#S(\ell_2^2; L_2) + \sum_{L_1 \le d \le L_2, \ell_2^2 \mid \varphi(d)} \frac{\ell_2 x \log\log x}{d^2} \right). \end{aligned} \tag{13}$$

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In this case, $\tau(\ell_2^2) \le \tau(\ell^2) \le a_r$. By Lemma 1, we have

$$\#U_r(x) \ll a_r \sum_{\ell_2 \le L_1} \left(\frac{L_2(c_1 \log\log x)^{\Omega(\ell_2)}}{\ell_2^2} + \frac{x(c_1 \log\log x)^{\Omega(\ell_2)+1}}{L_1 \ell_2} \right)$$
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Since $\ell_2 < L_2^{1/2}$,

$$\Omega(\ell_2^2) = 2\omega(\ell_2) < \frac{(1 + o(1)) \log L_2}{\log \log x}. \quad (16)$$

By Lemma 2, we have

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Inserting (16) and (17) into (15), we obtain

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Other problems

Among 38 almost Lehmer numbers below 2^{32}, 14 numbers are Carmichael and the others are not. Among five 1-Nearly Lehmer numbers below 2^{32}, only 1729 and 3069196417 are Carmichael. Are these numbers infinitely often Carmichael / non-Carmichael?

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MANY THANKS FOR YOUR ATTENTION

To be continued...?

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